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Invariant elastic constants and eigentensors of orthorhombic, tetragonal, hexagonal and cubic crystalline media

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The purpose of this paper is to present a simple and direct way of determining the eigenvalues and eigentensors, as well as their orientations, for all crystals of the orthorhombic, tetragonal, hexagonal and cubic symmetries, a procedure based on the spectral decomposition of the compliance and stiffness fourth-rank tensors. First, both the eigenvalues and the idempotent fourth-rank tensors are derived for the orthorhombic and tetragonal-7 symmetries. The latter decompose, respectively, the second-rank symmetric tensor spaces of orthorhombic and tetragonal-7 media into orthogonal subspaces, consisting of the stress and strain eigentensors, and split the elastic potential into distinct non-interacting strain-energy parts. Accordingly, the spectrum of the compliance tensor of the tetragonal-6 symmetry is evaluated, by reduction of the eigenvalues and eigentensors of either the orthorhombic or tetragonal-7 symmetry. These results are, then, applied in turn to each of the hexagonal and cubic crystal systems. In each case, the eigenvalues, the idempotent tensors and the stress and strain eigentensors are easily derived as particular cases of the results obtained for the tetragonal-6 symmetry. Furthermore, it is noted that the positivity of the eigenvalues for each symmetry is equivalent to the positive definiteness of the elastic potential and, thus, necessary and sufficient conditions are acquired, in terms of the compliance-tensor components, characteristic of each symmetry.

1. Introduction

Rychlewski (1984*a*,*b*) first employed the spectral decomposition theorem on the symmetric fourth-rank tensors of compliance **S** and stiffness **C** of a generic elastic solid, acting as linear operators on the symmetric second-rank-tensor space **L** of stresses $\boldsymbol{\sigma}$ and strains ε, for the decomposition of this space into orthogonal subspaces. Subsequently, the spectral formulation, which was initiated for the characterization of the theoretical algebraic structure of the linear elastic medium, was firmly incorporated by Theocaris & Philippidis (1989, 1990, 1991), and the extremely important and practical case of transversely isotropic media was treated in detail, emphasizing the role of eigenangles in the definition of the extent of anisotropy and toughness of such media. Recently, the spectral decomposition of the compliance tensor **S** was specifically defined for monoclinic media (Theocaris & Sokolis, 1999) as well as for orthotropic plates (Theocaris & Sokolis, 1998*a*,*b*).

Apart from the spectral decomposition of the compliance tensor **S**, many other types of decomposition have been proposed (Eshelby, 1961; Hill, 1965*a*,*b*; Srinivasan & Nigam, 1969; Willis, 1980; Walpole, 1981, 1984; Podio-Guidugli & Virga, 1987). Yet these types of decomposition, when compared with the spectral one, do not have its advantages; namely, the unparalleled simplicity that it introduces to the mathematical analysis of the theory of elasticity, a fact that is reflected in the elementary linear form that the generalized Hooke's law assumes.

In this paper, a simple and direct way is presented for the determination of the eigenvalues, the eigentensors and their orientations for anisotropic crystals. This procedure, based on the spectral decomposition of the compliance, **S**, and stiffness, **C**, fourth-rank tensors, is applied in turn to orthorhombic, tetragonal-7, tetragonal-6, hexagonal and cubic crystals (Fig. 1). By crystals, we here mean crystals or other non-crystalline anisotropic materials. Initially, the form of the compliance tensor **S** is recalled and its characteristic equation is solved. Thus, the corresponding explicit expressions for the eigenvalues are written down, and the idempotent fourth-rank tensors for the orthorhombic and tetragonal-7 symmetries are constructed, respectively, in §§2 and 3. These tensors decompose orthogonally the second-rank symmetric stress and strain tensors into their eigentensors, thus analyzing the total elastic strain-energy density of orthorhombic and tetragonal-7 crys-

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tals into independent elements. Next, the spectrum of the tetragonal-6 symmetry is evaluated by reduction of the spectrum of either the orthorhombic or the tetragonal-7 symmetry. Then, by regarding the crystalline media of the hexagonal and cubic systems as special tetragonal-6 crystals, the eigenvalues, the idempotent tensors and the stress and strain eigentensors, acquired for tetragonal-6 media, are determined successively in the case of hexagonal, in §4, and cubic symmetries, in §5. Moreover, it is shown in §6 that the positivity of the eigenvalues valid for each symmetry is equivalent to the positive definiteness of the elastic potential. Hence, necessary and sufficient conditions are deduced, in terms of the compliance-tensor components, representative of each distinct symmetry.

Finally, it is worthwhile mentioning that the analysis undertaken in this paper concerns the elastic compliance, **S**, and stiffness, **C**, fourth-rank tensors. Nonetheless, all the results presented herein remain valid for any fourth-rank tensor **H**, satisfying the full symmetry conditions $H_{ijkl} = H_{jikl} = H_{ijlk} = H_{klij}$.

2. Orthorhombic symmetry

In this section, we study the spectral decomposition of the compliance fourth-rank tensor **S** of an orthorhombic linear elastic crystal. The internal structure of such a medium is characterized by three axes of symmetry of the second order, L^2, and a plane of elastic symmetry, P. In this case, the Cartesian system, according to which the tensor components are referred, is oriented along the directions of the symmetry axes of the crystalline medium, with its 3 axis coinciding with one axis of elastic symmetry L^2 and its 1 axis lying either in the direction of another symmetry axis L^2 or normal to plane P. Hence, the compliance-tensor components S_{ijkl} of the orthorhombic system (Nye, 1957; Hearmon, 1961; Lekhnitskii, 1963) are expressed, in terms of the components s_{ij} of the 6×6 matrix **s** of the Voigt notation, as follows:

$$S_{1111} = s_{11}, \quad S_{2222} = s_{22}, \quad S_{3333} = s_{33}, \tag{1a}$$

$$S_{1122} = S_{2211} = s_{12}, \quad S_{2233} = S_{3322} = s_{23}, \quad S_{1133} = S_{3311} = s_{13}, \tag{1b}$$

$$S_{2323} = S_{2332} = S_{3223} = S_{3232} = \tfrac{1}{4}s_{44}, \tag{1c}$$

$$S_{1313} = S_{1331} = S_{3113} = S_{3131} = \tfrac{1}{4}s_{55}, \tag{1d}$$

$$S_{1212} = S_{1221} = S_{2112} = S_{2121} = \tfrac{1}{4}s_{66}. \tag{1e}$$

The eigenvalues λ_m of the associated square matrix of rank six to tensor **S** defined above were determined from the characteristic equation

$$\det\begin{bmatrix} s_{11}-\lambda & s_{12} & s_{13} & 0 & 0 & 0 \\ s_{12} & s_{22}-\lambda & s_{23} & 0 & 0 & 0 \\ s_{13} & s_{23} & s_{33}-\lambda & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2}s_{44}-\lambda & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2}s_{55}-\lambda & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2}s_{66}-\lambda \end{bmatrix} = 0, \tag{2}$$

which is equivalent to the following equation:

$$(\lambda^3 + A\lambda^2 + B\lambda + C)(\lambda - \tfrac{1}{2}s_{44})(\lambda - \tfrac{1}{2}s_{55})(\lambda - \tfrac{1}{2}s_{66}) = 0, \tag{3}$$

where the coefficients A, B and C are as follows:

$$A = -(s_{11} + s_{22} + s_{33}), \tag{4a}$$

$$B = (s_{11}s_{22} + s_{11}s_{33} + s_{22}s_{33}) - (s_{12}^2 + s_{13}^2 + s_{23}^2), \tag{4b}$$

$$C = s_{11}s_{23}^2 + s_{22}s_{13}^2 + s_{33}s_{12}^2 - (s_{11}s_{22}s_{33} + 2s_{12}s_{13}s_{23}). \tag{4c}$$

The polynomial inside the first parentheses of relation (3) is a cubic and, therefore, has to be transformed to its reduced form, in order for the set of eigenvalues λ_1, λ_2 and λ_3 to be estimated. Substituting for $\lambda = y - A/3$ in the cubic equation, this is recast as

$$y^3 + Py + Q = 0, \tag{5}$$

in terms of parameters $P = B - A^2/3$ and $Q = 2A^2/27 - AB/3 + C$. To solve (5), we first put $y = u\cos\theta$ and compare the resulting equation,

$$u^3\cos^3\theta + Pu\cos\theta + Q = 0, \tag{6}$$

with the trigonometric identity

$$4\cos^3\theta - 3\cos\theta = \cos 3\theta. \tag{7}$$

If these two equations are to be the same, we must have

$$\tfrac{1}{4}u^3 = ku = -Q/\cos 3\theta, \tag{8}$$

where $k = -P/3$. Hence, it is derived that

$$u = 0, \pm 2k^{1/2} \quad \text{and} \quad \cos 3\theta = -Q/ku. \tag{9}$$

Clearly, the case of $u = 0$ is to be excluded, since this requires $\cos 3\theta$ to exceed unity. With $u = 2k^{1/2}$, we find

$$\cos 3\theta = -Q/2k^{3/2} \tag{10a}$$

and, therefore,

$$\theta = \tfrac{2}{3}\pi n \pm \tfrac{1}{3}\cos^{-1}(-Q/2k^{3/2}), \quad n = 0, 1, 2, \ldots. \tag{10b}$$

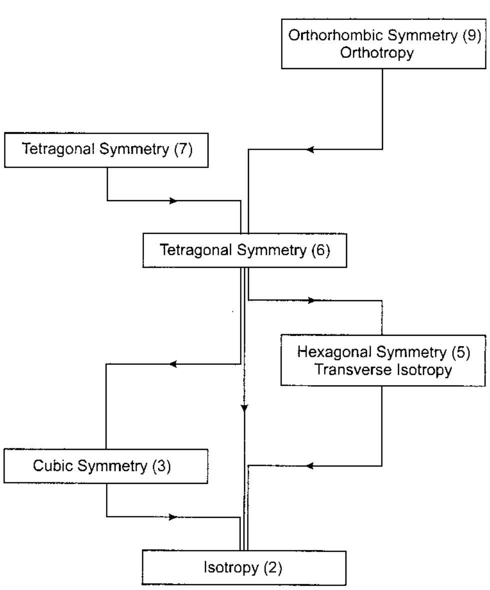

Figure 1
The relationship between the orthorhombic, tetragonal-7, tetragonal-6, hexagonal, cubic and isotropic symmetries.

Consequently, the eigenvalues λ_m, $m = 1, \ldots, 6$, of the compliance tensor **S** were evaluated to be of multiplicity 1, defined by

$$\lambda_1 = 2k^{1/2}\cos[\tfrac{1}{3}\cos^{-1}(-Q/2k^{3/2})] - A/3, \quad (11a)$$

$$\lambda_2 = 2k^{1/2}\cos\{\tfrac{1}{3}[\cos^{-1}(-Q/2k^{3/2}) + 2\pi]\} - A/3, \quad (11b)$$

$$\lambda_3 = 2k^{1/2}\cos\{\tfrac{1}{3}[\cos^{-1}(-Q/2k^{3/2}) + 4\pi]\} - A/3, \quad (11c)$$

$$\lambda_4 = \tfrac{1}{2}s_{44}, \quad \lambda_5 = \tfrac{1}{2}s_{55}, \quad \lambda_6 = \tfrac{1}{2}s_{66}, \quad (11d)$$

and it is easily verified that $u = -2k^{1/2}$ just reproduces the initial roots λ_1, λ_2 and λ_3.

The corresponding six idempotent fourth-rank tensors $\mathbf{E}_m$, $m = 1, \ldots, 6$, were also found as follows:

$$\mathbf{E}_1 = E^1_{ijkl} = \mathbf{h} \otimes \mathbf{h} = h_{ij}h_{kl}, \quad \mathbf{E}_2 = E^2_{ijkl} = \mathbf{j} \otimes \mathbf{j} = j_{ij}j_{kl}, \quad (12a)$$

$$\mathbf{E}_3 = E^3_{ijkl} = \mathbf{r} \otimes \mathbf{r} = r_{ij}r_{kl}, \quad \mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{h}, \mathbf{j}, \mathbf{r} \in \mathbf{L}, \quad (12b)$$

$$\mathbf{E}_4 = E^4_{ijkl} = \tfrac{1}{2}(a_{ik}b_{jl} + a_{il}b_{jk} + a_{jk}b_{il} + a_{jl}b_{ik}), \quad (12c)$$

$$\mathbf{E}_5 = E^5_{ijkl} = \tfrac{1}{2}(c_{ik}a_{jl} + c_{il}a_{jk} + c_{jk}a_{il} + c_{jl}a_{ik}), \quad (12d)$$

$$\mathbf{E}_6 = E^6_{ijkl} = \tfrac{1}{2}(b_{ik}c_{jl} + b_{il}c_{jk} + b_{jk}c_{il} + b_{jl}c_{ik}), \quad (12e)$$

in which the second-rank symmetric tensors **h**, **j** and **r**, appearing in the above-cited expressions for the idempotent tensors $\mathbf{E}_1$, $\mathbf{E}_2$ and $\mathbf{E}_3$, are given by

$$\mathbf{h} = \sin\theta\sin\varphi\mathbf{c} - \sin\theta\cos\varphi\mathbf{b} + \cos\theta\mathbf{a}, \quad (13a)$$

$$\mathbf{j} = -\sin\omega\mathbf{f} + \cos\omega\mathbf{g}, \quad \mathbf{r} = \cos\omega\mathbf{f} + \sin\omega\mathbf{g}, \quad (13b)$$

where tensors **f** and **g** are second-rank symmetric tensors, defined as follows:

$$\mathbf{f} = \cos\varphi\mathbf{c} + \sin\varphi\mathbf{b}, \quad (14a)$$

$$\mathbf{g} = -\cos\theta\sin\varphi\mathbf{c} + \cos\theta\cos\varphi\mathbf{b} + \sin\theta\mathbf{a}. \quad (14b)$$

Further, tensors **a**, **b** and **c**, figuring in relations (12*c*), (12*d*), (12*e*) for the idempotent tensors $\mathbf{E}_4$, $\mathbf{E}_5$ and $\mathbf{E}_6$, are second-rank symmetric tensors, defined by

$$\mathbf{a} = \mathbf{k} \otimes \mathbf{k}, \quad \mathbf{b} = \mathbf{l} \otimes \mathbf{l}, \quad \mathbf{c} = \mathbf{m} \otimes \mathbf{m}, \quad (15)$$

in which **k**, **l** and **m** constitute the unit vectors of the adopted Cartesian space $\mathbf{R}^3$, referring to the 3, 2 and 1 directions. Besides, the three angles θ, φ and ω of relations (13) and (14), referred to as the eigenangles, are expressed as

$$\tan\theta = \frac{[B_1^2 + (s_{11} - \lambda_1)^2 C_1^2/s_{11}^2]^{1/2}}{[(s_{11} - \lambda_1)A_1/s_{11}]}, \quad (16a)$$

$$\tan\varphi = \frac{[s_{12}s_{23} + s_{13}(s_{22} - \lambda_1)]}{[s_{12}s_{13} + s_{23}(s_{11} - \lambda_1)]}, \quad (16b)$$

$$\tan\omega = \left\{\frac{B_1^2 + (C_1^2/s_{11}^2)(s_{11} - \lambda_1)^2}{B_1^2 + [(A_1^2 + C_1^2)/s_{11}^2](s_{11} - \lambda_1)^2} - \frac{(A_2^2/s_{11}^2)(s_{11} - \lambda_2)^2}{\{B_2^2 + [(A_2^2 + C_2^2)/s_{11}^2](s_{11} - \lambda_2)^2\}}\right\}^{1/2} \times \left(\frac{(A_2/s_{11})(s_{11} - \lambda_2)}{\{B_2^2 + [(A_2^2 + C_2^2)/s_{11}^2](s_{11} - \lambda_2)^2\}^{1/2}}\right)^{-1}, \quad (16c)$$

with parameters A_i, B_i and C_i, $i = 1, 2$, which appear in the definitions above, given by

$$A_i = (s_{22} - \lambda_i)(s_{11} - \lambda_i) - s_{12}^2, \quad (17a)$$

$$B_i = (s_{11} - \lambda_i)[s_{12}s_{23} + s_{13}(s_{22} - \lambda_i)], \quad (17b)$$

$$C_i = s_{12}s_{13} + s_{23}(s_{11} - \lambda_i) \qquad i = 1, 2. \quad (17c)$$

In conclusion, the six eigenvalues λ_m, $m = 1, \ldots, 6$, together with the eigenangles θ, φ and ω, make up the nine quantities, essential for the coordinate-invariant description of the elastic characteristics of orthorhombic crystals. By means of the eigenvalues λ_m, $m = 1, \ldots, 6$, defined by relations (11), and the associated idempotent tensors $\mathbf{E}_m$, $m = 1, \ldots, 6$, given by relations (12), the compliance tensor **S** is analyzed as

$$\mathbf{S} = \lambda_1\mathbf{E}_1 + \ldots + \lambda_6\mathbf{E}_6 \quad (18)$$

and, thus, the space **L** of second-rank symmetric tensors is expanded orthogonally into six subspaces $\mathbf{L}_m$, $m = 1, \ldots, 6$:

$$\mathbf{L} = \mathbf{L}_1 \oplus \ldots \oplus \mathbf{L}_6, \quad (19)$$

where $\mathbf{L}_1$, $\mathbf{L}_2$ and $\mathbf{L}_3$ are one-dimensional subspaces, comprising combinations of deviatoric and hydrostatic tensors, whereas $\mathbf{L}_4$, $\mathbf{L}_5$ and $\mathbf{L}_6$ are one-dimensional subspaces of deviatoric tensors.

Next, the stress eigentensors $\overline{\boldsymbol{\sigma}_m}$ of the compliance fourth-rank tensor **S** of the orthorhombic medium are derived, by orthogonally projecting the second-rank symmetric stress tensor $\boldsymbol{\sigma}$ on subspaces $\mathbf{L}_m$, formed by the idempotent fourth-rank tensors $\mathbf{E}_m$, as follows:

$$\overline{\boldsymbol{\sigma}_m} = \mathbf{E}_m \cdot \boldsymbol{\sigma}, \quad m = 1, \ldots, 6. \quad (20)$$

Denoting by $\boldsymbol{\sigma}$ the contracted stress tensor, in the form of a 6D vector:

$$\boldsymbol{\sigma} = [\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_6]^T, \quad (21)$$

and carrying out the calculations, indicated by relations (20), the stress eigentensors $\overline{\boldsymbol{\sigma}_m}$ were found in contracted notation:

$$\overline{\boldsymbol{\sigma}_1} = [\sin\theta\sin\varphi(\sigma_1) - \sin\theta\cos\varphi(\sigma_2) + \cos\theta(\sigma_3)] \times [\sin\theta\sin\varphi, -\sin\theta\cos\varphi, \cos\theta, 0, 0, 0]^T, \quad (22a)$$

$$\overline{\boldsymbol{\sigma}_2} = [(-\cos\theta\sin\varphi\cos\omega - \cos\varphi\sin\omega)(\sigma_1) + (\cos\theta\cos\varphi\cos\omega - \sin\varphi\sin\omega)(\sigma_2) + \sin\theta\cos\omega(\sigma_3)] \times [-\cos\theta\sin\varphi\cos\omega - \cos\varphi\sin\omega, \cos\theta\cos\varphi\cos\omega - \sin\varphi\sin\omega, \sin\theta\cos\omega, 0, 0, 0]^T, \quad (22b)$$

$$\overline{\boldsymbol{\sigma}_3} = [(-\cos\theta\sin\varphi\sin\omega + \cos\varphi\cos\omega)(\sigma_1) + (\cos\theta\cos\varphi\sin\omega + \sin\varphi\cos\omega)(\sigma_2) + \sin\theta\sin\omega(\sigma_3)] \times [-\cos\theta\sin\varphi\sin\omega + \cos\varphi\cos\omega, \cos\theta\cos\varphi\sin\omega + \sin\varphi\cos\omega, \sin\theta\sin\omega, 0, 0, 0]^T, \quad (22c)$$

$$\overline{\boldsymbol{\sigma}_4} = [0, 0, 0, \sigma_4, 0, 0]^T, \quad (22d)$$

$$\overline{\boldsymbol{\sigma}_5} = [0, 0, 0, 0, \sigma_5, 0]^T, \quad (22e)$$

$$\overline{\boldsymbol{\sigma}_6} = [0, 0, 0, 0, 0, \sigma_6]^T. \quad (22f)$$

As may be noticed in relations (22), the stress eigentensors $\overline{\boldsymbol{\sigma}_m}$ split the stress tensor $\boldsymbol{\sigma}$ into six components, with tensors $\overline{\boldsymbol{\sigma}_1}$, $\overline{\boldsymbol{\sigma}_2}$ and $\overline{\boldsymbol{\sigma}_3}$ representing superposition of stressing along the 1, 2 and 3 directions of the adopted Cartesian coordinate system and tensors $\overline{\boldsymbol{\sigma}_4}$, $\overline{\boldsymbol{\sigma}_5}$ and $\overline{\boldsymbol{\sigma}_6}$ constituting simple shear stress states. Further, as can be observed, the contracted stress eigentensors $\overline{\boldsymbol{\sigma}_1}$, $\overline{\boldsymbol{\sigma}_2}$ and $\overline{\boldsymbol{\sigma}_3}$ are dependent on the values of the eigenangles θ, φ and ω, given by relations (16), and the elastic compliances of the crystal. Conversely, the remaining three characteristic stress tensors $\overline{\boldsymbol{\sigma}_4}$, $\overline{\boldsymbol{\sigma}_5}$ and $\overline{\boldsymbol{\sigma}_6}$ are independent of the eigenangles θ, φ and ω and the material properties, hence remaining the same for the whole class of orthorhombic media.

Next, according to the spectral decomposition of the compliance, **S**, and stress, $\boldsymbol{\sigma}$, tensors, the elastic potential T is divided,

$$2T(\boldsymbol{\sigma}) = T(\overline{\boldsymbol{\sigma}_1}) + \ldots + T(\overline{\boldsymbol{\sigma}_6}), \qquad (23)$$

into distinct energy constituents $T(\overline{\boldsymbol{\sigma}_m})$, each related solely to a single stress eigentensor $\overline{\boldsymbol{\sigma}_m}$ and defined by the following quantity:

$$T(\overline{\boldsymbol{\sigma}_m}) = \lambda_m(\overline{\boldsymbol{\sigma}_m} \cdot \overline{\boldsymbol{\sigma}_m}) = \lambda_m \operatorname{tr}(\overline{\boldsymbol{\sigma}_m}^2), \quad m = 1, \ldots, 6. \qquad (24)$$

In view of relations (24), the energy components $T(\overline{\boldsymbol{\sigma}_1})$, $T(\overline{\boldsymbol{\sigma}_2})$ and $T(\overline{\boldsymbol{\sigma}_3})$ are complex functions of the three eigenvalues λ_1, λ_2 and λ_3, as well as of the eigenangles θ, φ and ω. Hence, these are dependent on the values of the elastic compliances of the orthorhombic crystal and are associated with mixtures of dilatational and distortional forms of energy. On the other hand, the remaining elastic energy components, namely $T(\overline{\boldsymbol{\sigma}_4})$, $T(\overline{\boldsymbol{\sigma}_5})$ and $T(\overline{\boldsymbol{\sigma}_6})$, are independent of the values of the set of eigenangles θ, φ and ω and refer exclusively to shape distortion of the medium.

Nevertheless, in evaluating the above-mentioned elastic potential components $T(\overline{\boldsymbol{\sigma}_m})$, care must be taken to employ the full tensor notation for the stress eigentensors $\overline{\boldsymbol{\sigma}_m}$ and not their contracted notation expressed in equations (22) in terms of six-dimensional vectors, owing to the fact that the stress tensor $\boldsymbol{\sigma}$ has nine symmetric components and not six. Alternatively, the representation of the stress eigentensors $\overline{\boldsymbol{\sigma}_m}$ by vectors in a nine-dimensional space would be convenient, since their scalar product is then directly equal to the convolution of the stress eigentensors $\overline{\boldsymbol{\sigma}_m}$, defined in equation (24). Of course, it is also possible to consider the six-dimensional space and take as components of the stress vector the six distinct components of the stress tensor multiplied by certain numbers, the latter being chosen so that the scalar product of vectors will correspond to the convolution of tensors.

Lastly, it is of interest to note that, by considering the projections of the contracted stress eigentensors $\overline{\boldsymbol{\sigma}_m}$, $m = 1, \ldots, 6$, on the principal stress $(\sigma_1, \sigma_2, \sigma_3)$ frame, the $\overline{\boldsymbol{\sigma}_4}$, $\overline{\boldsymbol{\sigma}_5}$ and $\overline{\boldsymbol{\sigma}_6}$ eigentensors readily vanish. On the contrary, the characteristic stress states $\overline{\boldsymbol{\sigma}_1}$, $\overline{\boldsymbol{\sigma}_2}$ and $\overline{\boldsymbol{\sigma}_3}$ are represented by three mutually orthogonal vectors, oriented along the directions with the following associated unit vectors $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$:

$$\mathbf{e}_1 = [\sin\theta\sin\varphi, -\sin\theta\cos\varphi, \cos\theta]^T, \qquad (25a)$$

$$\mathbf{e}_2 = [-\cos\theta\sin\varphi\cos\omega - \cos\varphi\sin\omega, \\ \cos\theta\cos\varphi\cos\omega - \sin\varphi\sin\omega, \sin\theta\cos\omega]^T, \qquad (25b)$$

$$\mathbf{e}_3 = [-\cos\theta\sin\varphi\sin\omega + \cos\varphi\cos\omega, \\ \cos\theta\cos\varphi\sin\omega + \sin\varphi\cos\omega, \sin\theta\sin\omega]^T. \qquad (25c)$$

Fig. 2 exhibits the geometric arrangement of these three unit vectors, $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$, with respect to the principal stress $(\sigma_1, \sigma_2, \sigma_3)$ frame. Then, as noted in Fig. 2 and verified by relations (25), vectors $\mathbf{e}_2$ and $\mathbf{e}_3$ lie on plane (Θ', Θ), with vector $\mathbf{e}_2$ subtending an angle equal to ω with the Θ' axis and vector $\mathbf{e}_3$ subtending an angle equal to $(\pi/2 - \omega)$ with the same axis. In addition, the (Θ', Θ) plane is inclined to plane (σ_3, Θ) by an angle $(\pi/2 - \theta)$ and the Θ axis subtends an angle φ with the σ_1 axis. Finally, the $\mathbf{e}_1$ unit vector is perpendicular to the (Θ', Θ) plane, thus forming with the other two unit vectors, $\mathbf{e}_2$ and $\mathbf{e}_3$, a tri-orthogonal frame of vectors. This concludes the spectral decomposition of the compliance tensor **S** for the orthorhombic symmetry. For further discussion of some aspects and potential applications of the spectral analysis to orthotropic engineering materials, such as the fiber-reinforced composites, the reader may refer to a companion paper (Theocaris & Sokolis, 2000).

3. Tetragonal symmetry

3.1. Tetragonal-7 crystals

Consider the spectral decomposition of the compliance tensor **S** of a tetragonal-7 crystalline linear elastic medium. Such an anisotropic body is characterized by an axis of complex symmetry of the fourth order, L_4, or, alternatively, by an axis of symmetry of the second order, which is also an axis of complex symmetry of the fourth order, L_4^2. We suppose that the Cartesian coordinate system, to which the stress, $\boldsymbol{\sigma}$, and strain, ε, tensors refer, is aligned with the principal material directions, and its 3 axis is oriented along the axis of symmetry of the tetragonal-7 medium. Then, using the components s_{ij} of the 6×6 matrix **s** of the Voigt notation (Nye, 1957; Hearmon, 1961; Lekhnitskii, 1963), the compliance-tensor components S_{ijkl}, which are associated with the adopted reference system, are expressed as

$$S_{1111} = S_{2222} = s_{11}, \quad S_{3333} = s_{33}, \qquad (26a)$$

$$S_{1122} = S_{2211} = s_{12}, \quad S_{2233} = S_{3322} = S_{1133} = S_{3311} = s_{13}, \qquad (26b)$$

$$S_{2323} = S_{2332} = S_{3223} = S_{3232} = S_{1313} = S_{1331} = S_{3113} = S_{3131} \\ = \tfrac{1}{4}s_{44}, \qquad (26c)$$

$$S_{1212} = S_{1221} = S_{2112} = S_{2121} = \tfrac{1}{4}s_{66}, \qquad (26d)$$

$$S_{1121} = S_{1112} = S_{1211} = S_{2111} = \tfrac{1}{2}s_{16}, \qquad (26e)$$

$$S_{2212} = S_{2221} = S_{1222} = S_{2122} = -\tfrac{1}{2}s_{16}, \qquad (26f)$$

whereas the remaining components S_{ijkl} are equal to zero. The tensor **S** of such a medium, when spectrally decomposed, was shown to assume the following representation:

$$\mathbf{S} = \lambda_1\mathbf{E}_1 + \ldots + \lambda_5\mathbf{E}_5, \qquad (27)$$

where the characteristic values λ_m, $m = 1, \ldots, 5$, of the square matrix of rank six, associated with tensor **S**, were determined by solving its eigenvalue equation:

$$\det\begin{bmatrix} s_{11}-\lambda & s_{12} & s_{13} & 0 & 0 & s_{16}/2^{1/2} \\ s_{12} & s_{11}-\lambda & s_{13} & 0 & 0 & -s_{16}/2^{1/2} \\ s_{13} & s_{13} & s_{33}-\lambda & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2}s_{44}-\lambda & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2}s_{44}-\lambda & 0 \\ s_{16}/2^{1/2} & -s_{16}/2^{1/2} & 0 & 0 & 0 & \frac{1}{2}s_{66}-\lambda \end{bmatrix} = 0 \quad (28)$$

and were, thus, evaluated to be given by

$$\lambda_{1,5} = \left[\left(\frac{s_{11}-s_{12}}{2}\right) + \frac{s_{66}}{4}\right] \pm \left\{\left[\left(\frac{s_{11}-s_{12}}{2}\right) - \frac{s_{66}}{4}\right]^2 + s_{16}^2\right\}^{1/2}, \quad (29a)$$

$$\lambda_{2,3} = \left[\left(\frac{s_{11}+s_{12}}{2}\right) + \frac{s_{33}}{2}\right] \pm \left\{\left[\left(\frac{s_{11}+s_{12}}{2}\right) - \frac{s_{33}}{2}\right]^2 + 2s_{13}^2\right\}^{1/2}, \quad (29b)$$

$$\lambda_4 = s_{44}/2. \quad (29c)$$

Hence, there are five distinct eigenvalues, four of which, λ_1, λ_2, λ_3 and λ_5, are of multiplicity one, and the other, λ_4, is of multiplicity two. The associated five idempotent tensors $\mathbf{E}_m$, $m = 1, \ldots, 5$, of the spectral decomposition of **S** were also found as follows:

$$\mathbf{E}_1 = E^1_{ijkl} = \mathbf{h} \otimes \mathbf{h} = h_{ij}h_{kl}, \qquad \mathbf{E}_2 = E^2_{ijkl} = \mathbf{f} \otimes \mathbf{f} = f_{ij}f_{kl}, \quad (30a)$$

$$\mathbf{E}_3 = E^3_{ijkl} = \mathbf{g} \otimes \mathbf{g} = g_{ij}g_{kl}, \qquad \mathbf{E}_5 = E^5_{ijkl} = \mathbf{r} \otimes \mathbf{r} = r_{ij}r_{kl}, \quad (30b)$$

$$\mathbf{E}_4 = E^4_{ijkl} = \tfrac{1}{2}[(k_i m_j + m_i k_j)(k_k m_l + m_k k_l) + (k_i l_j + l_i k_j)(k_k l_l + l_k k_l)]. \quad (30c)$$

The second-rank axisymmetric tensors **h**, **f**, **g** and **r** of relations (30) depend on the components of the compliance tensor **S**. They are defined by

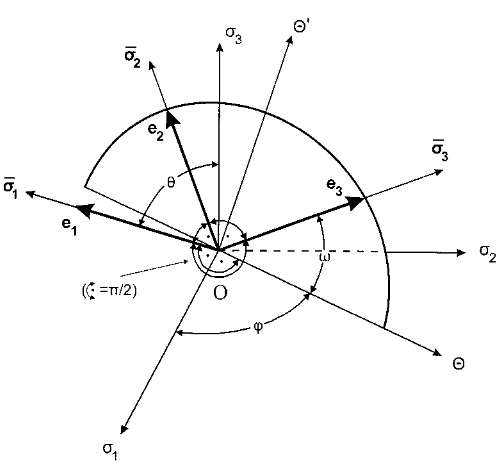

Figure 2
Geometric representation of the contracted stress eigentensors of the compliance fourth-rank tensor for orthorhombic crystalline media in the principal stress system ($\sigma_1, \sigma_2, \sigma_3$).

$$\mathbf{h} = 2^{-1/2}\cos\rho(-\mathbf{b}+\mathbf{c}) + \sin\rho\mathbf{d}, \quad (31a)$$

$$\mathbf{f} = -2^{-1/2}\sin\omega(\mathbf{b}+\mathbf{c}) + \cos\omega\mathbf{a}, \quad (31b)$$

$$\mathbf{g} = 2^{-1/2}\cos\omega(\mathbf{b}+\mathbf{c}) + \sin\omega\mathbf{a}, \quad (31c)$$

$$\mathbf{r} = -2^{-1/2}\sin\rho(-\mathbf{b}+\mathbf{c}) + \cos\rho\mathbf{d}, \quad (31d)$$

with the second-rank symmetric tensor **d** expressed as

$$\mathbf{d} = 2^{-1/2}(\mathbf{l} \otimes \mathbf{m} + \mathbf{m} \otimes \mathbf{l}). \quad (32)$$

Angles ω and ρ, cited as the eigenangles, are defined as follows:

$$\tan 2\omega = \frac{2 \times 2^{1/2}s_{13}}{[(s_{11}+s_{12}) - s_{33}]}, \qquad \tan 2\rho = \frac{2s_{16}}{[(s_{11}-s_{12}) - s_{66}/2]}. \quad (33)$$

It is concluded that the space **L** of second-rank symmetric tensors of tetragonal media is decomposed into five subspaces $\mathbf{L}_m$:

$$\mathbf{L} = \mathbf{L}_1 \oplus \ldots \oplus \mathbf{L}_5, \quad (34)$$

and it is deduced that an invariant description of the elasticity of crystalline or other anisotropic media of the tetragonal system is offered by means of the five different eigenvalues λ_m, $m = 1, \ldots, 5$, of the compliance tensor **S**, together with the values of the two eigenangles ω and ρ.

In addition, the second-rank stress eigentensors $\overline{\boldsymbol{\sigma}_m}$, $m = 1, \ldots, 5$, of the compliance fourth-rank tensor **S** presenting the tetragonal symmetry are derived, by carrying out the series of computations implicit in relations (20). These are expressed in contracted notation as

$$\overline{\boldsymbol{\sigma}_1} = [2^{-1/2}\cos\rho(\sigma_1 - \sigma_2) + \sin\rho(\sigma_6)] \times [2^{-1/2}\cos\rho, -2^{-1/2}\cos\rho, 0, 0, 0, \sin\rho]^T, \quad (35a)$$

$$\overline{\boldsymbol{\sigma}_2} = [-2^{-1/2}\sin\omega(\sigma_1 + \sigma_2) + \cos\omega(\sigma_3)] \times [-2^{-1/2}\sin\omega, -2^{-1/2}\sin\omega, \cos\omega, 0, 0, 0]^T, \quad (35b)$$

$$\overline{\boldsymbol{\sigma}_3} = [2^{-1/2}\cos\omega(\sigma_1 + \sigma_2) + \sin\omega(\sigma_3)] \times [2^{-1/2}\cos\omega, 2^{-1/2}\cos\omega, \sin\omega, 0, 0, 0]^T, \quad (35c)$$

$$\overline{\boldsymbol{\sigma}_4} = [0, 0, 0, \sigma_4, \sigma_5, 0]^T, \quad (35d)$$

$$\overline{\boldsymbol{\sigma}_5} = [-2^{-1/2}\sin\rho(\sigma_1 - \sigma_2) + \cos\rho(\sigma_6)] \times [-2^{-1/2}\sin\rho, 2^{-1/2}\sin\rho, 0, 0, 0, \cos\rho]^T. \quad (35e)$$

Relations (35) suggest that three of the characteristic stress tensors, $\overline{\boldsymbol{\sigma}_1}$, $\overline{\boldsymbol{\sigma}_4}$ and $\overline{\boldsymbol{\sigma}_5}$, are shears. Tensors $\overline{\boldsymbol{\sigma}_1}$ and $\overline{\boldsymbol{\sigma}_5}$ are superpositions of pure and simple shear, whereas $\overline{\boldsymbol{\sigma}_4}$ constitutes a simple shear loading. Moreover, the characteristic stress $\overline{\boldsymbol{\sigma}_2}$ and $\overline{\boldsymbol{\sigma}_3}$ tensors represent equilateral stressings in the 12 plane, together with tension or compression along the 3 axis of the adopted Cartesian coordinate system. Likewise, the $\overline{\boldsymbol{\sigma}_1}$ and $\overline{\boldsymbol{\sigma}_5}$ eigenstates are stresses in the 12 plane, as well as tension along the 12 axis. Besides, the characteristic stress tensors $\overline{\boldsymbol{\sigma}_1}$, $\overline{\boldsymbol{\sigma}_2}$, $\overline{\boldsymbol{\sigma}_3}$ and $\overline{\boldsymbol{\sigma}_5}$ are dependent on the values of the elastic constants of the tetragonal medium, with the components of the $\overline{\boldsymbol{\sigma}_1}$ and $\overline{\boldsymbol{\sigma}_5}$ eigenstates being functions of the eigenangle ρ, and the $\overline{\boldsymbol{\sigma}_2}$ and $\overline{\boldsymbol{\sigma}_3}$ eigentensors being functions of the eigenangle ω. Finally, the orthogonality of the stress eigentensors $\overline{\boldsymbol{\sigma}_m}$ to each other is easy to verify, as it is to verify that the sum of the five eigentensors is the stress tensor $\boldsymbol{\sigma}$.

It is noted that to the characteristic stress tensors $\overline{\boldsymbol{\sigma}_m}$ correspond strain tensors $\overline{\varepsilon_m}$, decomposing the total elastic strain-energy density T in discrete components $T(\overline{\boldsymbol{\sigma}_m})$, $m = 1, \ldots, 5$, in terms of equation (24). Their expressions are as follows:

$$\begin{aligned} T(\overline{\boldsymbol{\sigma}_1}) = \Big[\{[(s_{11} - s_{12})/2] + s_{66}/4\} \\ + \big(\{[(s_{11} - s_{12})/2] - s_{66}/4\}^2 + s_{16}^2\big)^{1/2}\Big] \\ \times [2^{-1/2}\cos\rho(\sigma_1 - \sigma_2) + 2^{1/2}\sin\rho(\sigma_6)]^2, \end{aligned} \qquad (36a)$$

$$\begin{aligned} T(\overline{\boldsymbol{\sigma}_2}) = \Big[\{[(s_{11} + s_{12})/2] + s_{33}/2\} \\ + \big(\{[(s_{11} + s_{12})/2] - s_{33}/4\}^2 + 2s_{13}^2\big)^{1/2}\Big] \\ \times [-2^{-1/2}\sin\omega(\sigma_1 + \sigma_2) + \cos\omega(\sigma_3)]^2, \end{aligned} \qquad (36b)$$

$$\begin{aligned} T(\overline{\boldsymbol{\sigma}_3}) = \Big[\{[(s_{11} + s_{12})/2] + s_{33}/2\} \\ - \big(\{[(s_{11} + s_{12})/2] - s_{33}/4\}^2 + 2s_{13}^2\big)^{1/2}\Big] \\ \times [2^{-1/2}\cos\omega(\sigma_1 + \sigma_2) + \sin\omega(\sigma_3)]^2, \end{aligned} \qquad (36c)$$

$$T(\overline{\boldsymbol{\sigma}_4}) = s_{44}(\sigma_4^2 + \sigma_5^2), \qquad (36d)$$

$$\begin{aligned} T(\overline{\boldsymbol{\sigma}_5}) = \Big[\{[(s_{11} - s_{12})/2] + s_{66}/4\} \\ - \big(\{[(s_{11} - s_{12})/2] - s_{66}/4\}^2 + s_{16}^2\big)^{1/2}\Big] \\ \times [-2^{-1/2}\sin\rho(\sigma_1 - \sigma_2) + 2^{1/2}\cos\rho(\sigma_6)]^2. \end{aligned} \qquad (36e)$$

It is implied by relations (36) that the $T(\overline{\boldsymbol{\sigma}_4})$ strain energy is independent of the values of the eigenangles ρ and ω and is solely associated with the distortional type of strain energy. Nevertheless, the remaining stress-energy elements, namely $T(\overline{\boldsymbol{\sigma}_1})$, $T(\overline{\boldsymbol{\sigma}_2})$, $T(\overline{\boldsymbol{\sigma}_3})$ and $T(\overline{\boldsymbol{\sigma}_5})$, correspond to mixtures of distortional and dilatational components of elastic energy. These depend on the values of the eigenangles ρ and ω, which, therefore, influence the type of energy formed for each particular tetragonal medium.

In addition, if we project the stress eigentensors $\overline{\boldsymbol{\sigma}_m}$, $m = 1, \ldots, 5$, on the principal stress space $(\sigma_1, \sigma_2, \sigma_3)$, it can readily be derived that the projection of eigentensor $\overline{\boldsymbol{\sigma}_4}$, which constitutes a simple shear loading, becomes equal to zero. On the contrary, the projections of the stress eigentensors $\overline{\boldsymbol{\sigma}_2}$, $\overline{\boldsymbol{\sigma}_3}$ and the sum of $\overline{\boldsymbol{\sigma}_1}$ and $\overline{\boldsymbol{\sigma}_5}$ are generally represented by a tri-orthogonal frame of vectors. These are associated with the respective unit vectors $\mathbf{e}_{15}$, $\mathbf{e}_2$ and $\mathbf{e}_3$, which are defined as follows (Fig. 3):

$$\mathbf{e}_{15} = [2^{-1/2}, -2^{-1/2}, 0]^T, \qquad (37a)$$

$$\mathbf{e}_2 = [-2^{-1/2}\sin\omega, -2^{-1/2}\sin\omega, \cos\omega]^T, \qquad (37b)$$

$$\mathbf{e}_3 = [2^{-1/2}\cos\omega, 2^{-1/2}\cos\omega, \sin\omega]^T. \qquad (37c)$$

According to relations (37), the unit vectors $\mathbf{e}_2$ and $\mathbf{e}_3$ are equally inclined to axes σ_1 and σ_2, thus lying on the principal diagonal plane (σ_3, δ_{12}) containing the σ_3 axis and passing through the bisector δ_{12} of $\angle\sigma_1 O\sigma_2$. In addition, vectors $\mathbf{e}_2$ and $\mathbf{e}_3$ subtend angles equal to ω and $(\pi/2 - \omega)$ with respect to the σ_3 axis, whereas vector $\mathbf{e}_1$ is perpendicular to the σ_3 axis and hence lies on the intersection of the deviatoric π plane and the plane $\sigma_3 = 0$.

Similarly, a geometric interpretation of the eigenangle ρ arises if one considers the projections of the characteristic stress tensors $\overline{\boldsymbol{\sigma}_m}$, $m = 1, \ldots, 5$, given by relations (35), on the stress system $(\sigma_1, \sigma_2, \sigma_6)$. Then, it is easily deduced that the projection of the $\overline{\boldsymbol{\sigma}_4}$ tensor vanishes, whereas the projections of the stress states $\overline{\boldsymbol{\sigma}_1}$, $\overline{\boldsymbol{\sigma}_5}$ and the sum of $\overline{\boldsymbol{\sigma}_2}$ and $\overline{\boldsymbol{\sigma}_3}$ are generally represented by three mutually orthogonal vectors, oriented along directions with the following associated unit vectors $\mathbf{e}_1$, $\mathbf{e}_{23}$ and $\mathbf{e}_5$:

$$\mathbf{e}_1 = [2^{-1/2}\cos\rho, -2^{-1/2}\cos\rho, \sin\rho]^T, \qquad (38a)$$

$$\mathbf{e}_{23} = [2^{-1/2}, 2^{-1/2}, 0]^T, \qquad (38b)$$

$$\mathbf{e}_5 = [-2^{-1/2}\sin\rho, 2^{-1/2}\sin\rho, \cos\rho]^T. \qquad (38c)$$

Fig. 4 presents the geometric arrangement of these three vectors, $\mathbf{e}_1$, $\mathbf{e}_{23}$ and $\mathbf{e}_5$. It may be inferred from relations (38) that the unit vectors $\mathbf{e}_1$ and $\mathbf{e}_5$ are equally inclined with respect to the principal stress axes σ_1 and $-\sigma_2$. As a result, they lie on the plane containing the σ_6 axis and passing through the line $\sigma_1 = -\sigma_2$. Furthermore, vectors $\mathbf{e}_1$ and $\mathbf{e}_5$, mutually orthogonal, subtend angles equal to ρ and $(\pi/2 - \rho)$ with the σ_6 axis. Vector $\mathbf{e}_{23}$ is normal to the same axis and to the plane $\sigma_1 = -\sigma_2$ and hence lies perpendicular to the intersection of this plane and the plane $\sigma_6 = 0$. This is valid for all tetragonal media, since the direction cosines of vector $\mathbf{e}_{23}$ are independent of the value of the eigenangle ρ and, therefore, of the elastic constants of the medium.

3.2. Tetragonal-6 crystals

Up to this point, all the discussion has been restricted to tetragonal-7 media, which are characterized by seven different elastic compliances s_{11}, s_{33}, s_{44}, s_{66}, s_{12}, s_{13}, s_{16}, defined by relations (26). However, tetragonal-6 crystals may also be found in the literature (Landolt-Bornstein, 1979, 1984). These are identified by six generally different non-zero elastic compliances, s_{11}, s_{33}, s_{44}, s_{66}, s_{12}, s_{13}, and exhibit either a

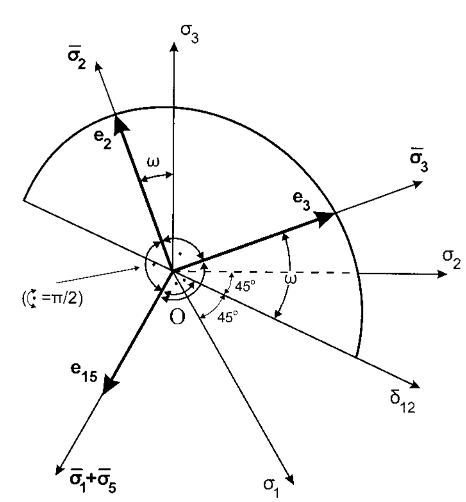

Figure 3
Projection of the characteristic stress states of the compliance fourth-rank tensor valid for tetragonal media in the principal stress frame $(\sigma_1, \sigma_2, \sigma_3)$.

symmetry axis of the fourth order, L^4, or an axis of symmetry of the second order, which is also an axis of complex symmetry of the fourth order, L_4^2. In addition, their crystallographic form possesses a symmetry axis of the second order, L^2, or an axis normal to a plane of symmetry P. It is assumed that the coordinate system has its 3 axis coinciding with L^4 or L_4^2 and the 1 axis coincides with L^2 or is normal to P. Accordingly, the eigenvalues λ_m, $m = 1, \ldots, 5$, of the compliance tensor $\mathbf{S}$ for tetragonal-6 crystals are obtained, either by regarding these media as special tetragonal-7 crystals and setting $s_{16} = 0$ or, alternatively, by considering them as particular orthorhombic ones under the conditions $s_{22} = s_{33}$, $s_{13} = s_{23}$ and $s_{44} = s_{55}$. These are expressed by

$$\lambda_1 = s_{11} - s_{12}, \quad \lambda_4 = s_{44}/2, \quad \lambda_5 = s_{66}/2, \tag{39a}$$

$$\lambda_{2,3} = \left[\left(\frac{s_{11} + s_{12}}{2}\right) + \frac{s_{33}}{2}\right] \pm \left\{\left[\left(\frac{s_{11} + s_{12}}{2}\right) - \frac{s_{33}}{2}\right]^2 + 2s_{13}^2\right\}^{1/2}. \tag{39b}$$

Again, there are five distinct eigenvalues λ_m. The second, third and fourth eigenvalues, *i.e.* λ_2, λ_3 and λ_4, are identical to those of tetragonal-7 symmetry, defined by relations (29*b*), (29*c*). The remaining two, λ_1 and λ_5, reduce upon substitution of $s_{16} = 0$ to $s_{11} - s_{12}$ and $s_{66}/2$, respectively. Moreover, the fourth and fifth eigenvalues of tetragonal-6 symmetry, that is λ_4 and λ_5, are identical to eigenvalues λ_4 and λ_6 of the orthorhombic symmetry, defined by equations (11*d*), whereas the remaining characteristic values are derived from the corresponding ones of the orthorhombic symmetry, with the equations of some compliances, $s_{22} = s_{33}$, $s_{13} = s_{23}$ and $s_{44} = s_{55}$. Specifically, parameters Q and k are expressed, under these conditions for the tetragonal-6 symmetry, as

$$Q = \tfrac{2}{27}(s_{11}^3 - s_{33}^3) - \tfrac{2}{3}(s_{11} - s_{33})[(s_{12}^2 - s_{13}^2) + (s_{11}s_{33})/3] - 2s_{12}s_{13}^2, \tag{40a}$$

$$k = \tfrac{1}{3}[(s_{11} - s_{33})^2/3 + (s_{12}^2 + 2s_{13}^2)]. \tag{40b}$$

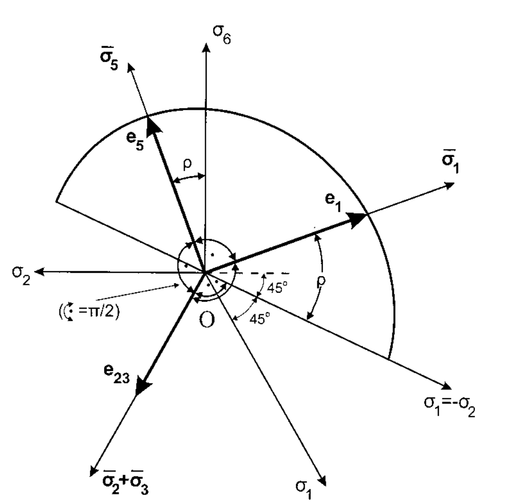

Figure 4
Projection of the characteristic stress states of the compliance fourth-rank tensor valid for tetragonal media in the stress frame (σ_1, σ_2, σ_6).

In fact, it is proven that Q and k are interrelated in terms of the ensuing relations:

$$\tfrac{1}{2}Q + \tfrac{1}{2}\{[(s_{11} - s_{33})/2] - s_{12}\}^3 = \tfrac{3}{2}k\{[(s_{11} - s_{33})/3] - s_{12}\}, \tag{41a}$$

$$\begin{aligned}\tfrac{1}{2}Q + \tfrac{1}{2}\big[&\{[(-s_{11} + 3s_{12})/6] + s_{33}/6\} \\ &\pm \big(\{[(s_{11} + s_{12})/2] - s_{33}/2\}^2 + 2s_{13}^2\big)^{1/2}\big]^3 \\ &= \tfrac{3}{2}k\big[\{[(-s_{11} + 3s_{12})/6] + s_{33}/6\} \\ &\qquad \pm \big(\{[(s_{11} + s_{12})/2] - s_{33}/2\}^2 + 2s_{13}^2\big)^{1/2}\big],\end{aligned} \tag{41b}$$

so that the ratio $-Q/2k^{3/2}$ is given by

$$\begin{aligned}&-Q/2k^{3/2} \\ &\quad = \cos\big[3\cos^{-1}\big(\{[(s_{11} - s_{33})/3] - s_{12}\}/2k^{1/2}\big) - 2\pi n\big] \\ &\quad = \cos\big(3\cos^{-1}\big\{\big[\{[(-s_{11} + 3s_{12})/6] + s_{33}/6\} \\ &\qquad \pm \big(\{[(s_{11} + s_{12})/2] - s_{33}/2\}^2 + 2s_{13}^2\big)\big]/2k^{1/2}\big\} - 2\pi n\big),\end{aligned} \tag{42}$$

where index n acquires the values 0, 1 and 2. In view of relation (42), it readily follows that the characteristic values λ_1, λ_2 and λ_3 of the orthorhombic symmetry, expressed by equations (11), degenerate for tetragonal-6 media into the forms given in relations (39). Yet each eigenvalue of the orthorhombic system does not correspond distinctly to every eigenvalue of the tetragonal-6 system.

Additionally, since the elastic constant s_{16} vanishes, the eigenangle ρ of tetragonal-6 media vanishes, whereas the eigenangle ω is defined by relation (33), as in the case of tetragonal-7 crystals. Further, the eigenangles θ and φ of orthotropy are equal to $\pi/2$ and $\pi/4$ for tetragonal-6 media, while the eigenangle ω reduces, upon substitution of the conditions for tetragonal-6 symmetry and the values of the respective eigenvalues of relations (39), to the definition of ω given in equation (33).

Besides, there are five distinct elementary idempotent tensors $\mathbf{E}_m$, $m = 1, \ldots, 5$, with the second, third and fourth, *i.e.* $\mathbf{E}_2$, $\mathbf{E}_3$ and $\mathbf{E}_4$, being identical to those of tetragonal-7 media. On the contrary, the idempotent tensors $\mathbf{E}_1$ and $\mathbf{E}_5$ are acquired by setting the value of the eigenangle ρ equal to zero. Therefore, the associated five idempotent tensors $\mathbf{E}_m$ are expressed by relations (30), with tensors $\mathbf{f}$ and $\mathbf{g}$ defined in terms of relations (31), and tensors $\mathbf{h}$ and $\mathbf{r}$ expressed as

$$\mathbf{h} = 2^{-1/2}(-\mathbf{b} + \mathbf{c}), \quad \mathbf{r} = 2^{1/2}\mathbf{d}. \tag{43}$$

Otherwise, the idempotent fourth-rank tensors $\mathbf{E}_m$ of tetragonal-6 media may be acquired from the corresponding ones of the orthorhombic medium, by substituting in expressions (12) the values of the eigenangles θ and φ for tetragonal-6 symmetry, which are equal to $\pi/2$ and $\pi/4$, respectively.

Finally, the corresponding stress eigentensors $\overline{\boldsymbol{\sigma}_m}$, $m = 1, \ldots, 5$, are given by

$$\overline{\boldsymbol{\sigma}_1} = [\tfrac{1}{2}(\sigma_1 - \sigma_2), \tfrac{1}{2}(\sigma_2 - \sigma_1), 0, 0, 0, 0]^T, \quad (44a)$$

$$\overline{\boldsymbol{\sigma}_2} = [-2^{-1/2}\sin\omega(\sigma_1 + \sigma_2) + \cos\omega(\sigma_3)] \times [-2^{-1/2}\sin\omega, -2^{-1/2}\sin\omega, \cos\omega, 0, 0, 0]^T, \quad (44b)$$

$$\overline{\boldsymbol{\sigma}_3} = [2^{-1/2}\cos\omega(\sigma_1 + \sigma_2) + \sin\omega(\sigma_3)] \times [2^{-1/2}\cos\omega, 2^{-1/2}\cos\omega, \sin\omega, 0, 0, 0]^T, \quad (44c)$$

$$\overline{\boldsymbol{\sigma}_4} = [0, 0, 0, \sigma_4, \sigma_5, 0]^T, \quad (44d)$$

$$\overline{\boldsymbol{\sigma}_5} = [0, 0, 0, 0, 0, \sigma_6]^T. \quad (44e)$$

Relations (44*a*), (44*d*) and (44*e*) indicate that the stress eigenstates $\overline{\boldsymbol{\sigma}_1}$, $\overline{\boldsymbol{\sigma}_4}$ and $\overline{\boldsymbol{\sigma}_5}$, related to the spectral decomposition of the compliance tensor **S** for tetragonal-6 media, are shear loadings, with the $\overline{\boldsymbol{\sigma}_1}$ eigentensor representing pure shear and states $\overline{\boldsymbol{\sigma}_4}$ and $\overline{\boldsymbol{\sigma}_5}$ constituting simple shear loadings. Therefore, these stress states correspond to strain states, producing shape alterations of the medium, and the strain energy, formed by these tensors, is associated solely with shape distortions of the tetragonal-6 medium.

4. Hexagonal symmetry

Hexagonal crystals are distinguished by five different non-zero elastic compliances s_{11}, s_{33}, s_{44}, s_{12}, s_{13}. Below, they are regarded as special tetragonal-6 crystals and the components S_{ijkl} of their compliance fourth-rank tensor **S** are defined by relations (26), with respect to the matrix components s_{ij} of the Voigt notation, setting $s_{16} = 0$ and $2s_{66} = (s_{11} - s_{12})$. Next, it is assumed that the Cartesian frame of reference is oriented, with its 3 axis being an axis of symmetry of either the third-order L^3 or the sixth-order L^6 and, if a symmetry axis of the second-order L^2 exists, this is taken to be the 1 axis. The eigenvalues λ_m, $m = 1, \ldots, 4$, of the compliance tensor **S** for hexagonal crystals were obtained, by reduction of the defining relations (39) for the eigenvalues λ_m of the tetragonal-6 medium. These are given by the following relations:

$$\lambda_1 = s_{11} - s_{12}, \quad \lambda_4 = s_{44}/2, \quad (45a)$$

$$\lambda_{2,3} = \left[\left(\frac{s_{11} + s_{12}}{2}\right) + \frac{s_{33}}{2}\right] \pm \left\{\left[\left(\frac{s_{11} + s_{12}}{2}\right) - \frac{s_{33}}{2}\right]^2 + 2s_{13}^2\right\}^{1/2}. \quad (45b)$$

All four eigenvalues λ_m of the hexagonal symmetry are identical to those of the tetragonal-6 symmetry. In addition, eigenvalue λ_5 of tetragonal-6 media, defined by relation (39*a*), degenerates to $s_{11} - s_{12}$. Then, there are four distinct characteristic values λ_m of tensor **S**; two of them, λ_1 and λ_4, are of multiplicity two, and the other two, λ_2 and λ_3, are of multiplicity one.

The value of the eigenangle ω is given for the hexagonal symmetry by relation (33), as in the case of the tetragonal-6 and -7 symmetries. In addition, the corresponding four idempotent tensors $\mathbf{E}_m$, $m = 1, \ldots, 4$, were determined from equations (30):

$$\mathbf{E}_1 = E^1_{ijkl} = \tfrac{1}{2}(b'_{ik}b'_{jl} + b'_{il}b'_{jk} - b'_{ij}b'_{kl}), \quad (46a)$$

$$\mathbf{E}_2 = E^2_{ijkl} = \mathbf{f} \otimes \mathbf{f} = f_{ij}f_{kl}, \quad (46b)$$

$$\mathbf{E}_3 = E^3_{ijkl} = \mathbf{g} \otimes \mathbf{g} = g_{ij}g_{kl}, \quad (46c)$$

$$\mathbf{E}_4 = E^4_{ijkl} = \tfrac{1}{2}(b'_{ik}a_{jl} + b'_{il}a_{jk} + b'_{jl}a_{ik} + b'_{jk}a_{il}). \quad (46d)$$

The second-rank symmetric tensor $\mathbf{b}'$, figuring in relations (46) for the idempotent tensors $\mathbf{E}_1$ and $\mathbf{E}_4$, is defined by $\mathbf{b}' = \mathbf{b} + \mathbf{c}$ and the second-rank symmetric tensors **b** and **c** are expressed in terms of relations (15). It is noted that the second, $\mathbf{E}_2$, third, $\mathbf{E}_3$, and fourth, $\mathbf{E}_4$, idempotent tensors of relations (46*b*), (46*c*) and (46*d*) are identical to those of tetragonal-6 symmetry, expressed by relations (30). On the contrary, the first, $\mathbf{E}_1$, and fifth, $\mathbf{E}_5$, idempotent tensors of tetragonal-6 media merge and degenerate to expression (46*a*) for the hexagonal symmetry.

Therefore, it is understood that the eigenvalues λ_m, $m = 1, \ldots, 4$, given by relations (45), and the eigenangle ω constitute the five invariant elastic constants, which are responsible for the description of the elastic features of hexagonal anisotropic media. Furthermore, for the characteristic values λ_m and the corresponding idempotent fourth-rank tensors $\mathbf{E}_m$, it is valid that

$$\mathbf{S} = \lambda_1\mathbf{E}_1 + \ldots + \lambda_4\mathbf{E}_4 \quad (47)$$

and, hence, the second-rank tensor space **L** is expanded orthogonally as

$$\mathbf{L} = \mathbf{L}_1 \oplus \ldots \oplus \mathbf{L}_4, \quad (48)$$

where $\mathbf{L}_1$ and $\mathbf{L}_4$ are two-dimensional subspaces of deviators and $\mathbf{L}_2$, $\mathbf{L}_3$ are one-dimensional subspaces.

Furthermore, the stress eigentensors $\overline{\boldsymbol{\sigma}_m}$, $m = 1, \ldots, 4$, of the compliance tensor **S** were derived *via* relations (44). These are given in contracted notation by

$$\overline{\boldsymbol{\sigma}_1} = [\tfrac{1}{2}(\sigma_1 - \sigma_2), \tfrac{1}{2}(\sigma_2 - \sigma_1), 0, 0, 0, \sigma_6]^T, \quad (49a)$$

$$\overline{\boldsymbol{\sigma}_2} = [-2^{-1/2}\sin\omega(\sigma_1 + \sigma_2) + \cos\omega(\sigma_3)] \times [-2^{-1/2}\sin\omega, -2^{-1/2}\sin\omega, \cos\omega, 0, 0, 0]^T, \quad (49b)$$

$$\overline{\boldsymbol{\sigma}_3} = [2^{-1/2}\cos\omega(\sigma_1 + \sigma_2) + \sin\omega(\sigma_3)] \times [2^{-1/2}\cos\omega, 2^{-1/2}\cos\omega, \sin\omega, 0, 0, 0]^T, \quad (49c)$$

$$\overline{\boldsymbol{\sigma}_4} = [0, 0, 0, \sigma_4, \sigma_5, 0]^T. \quad (49d)$$

It is easily verified from relations (49) that the sum of the four stress eigentensors $\overline{\boldsymbol{\sigma}_m}$, associated with the spectral decomposition of the compliance tensor **S** for hexagonal media, is the generic stress tensor $\boldsymbol{\sigma}$. The orthogonality of each of the stress eigenstates $\overline{\boldsymbol{\sigma}_m}$ to each of the other eigentensors in the set is also readily verified. Furthermore, it is noted that the $\overline{\boldsymbol{\sigma}_1}$ and $\overline{\boldsymbol{\sigma}_4}$ eigentensors do not rely on the elastic properties, thus remaining constant for the whole class of hexagonal media. On the other hand, the characteristic stress tensors $\overline{\boldsymbol{\sigma}_2}$ and $\overline{\boldsymbol{\sigma}_3}$ are determined by the value of the eigenangle ω, given by relation (33), and depend on the engineering elastic constants of the hexagonal body. In addition, the characteristic stress states $\overline{\boldsymbol{\sigma}_1}$ and $\overline{\boldsymbol{\sigma}_4}$ are shears, with $\overline{\boldsymbol{\sigma}_1}$ being a superposition of pure and simple shear and $\overline{\boldsymbol{\sigma}_4}$ being simple shear. The sum of eigentensors $\overline{\boldsymbol{\sigma}_2}$ and $\overline{\boldsymbol{\sigma}_3}$ is the orthogonal complement to the sum of the shear stress states $\overline{\boldsymbol{\sigma}_1}$ and $\overline{\boldsymbol{\sigma}_4}$. Finally, the $\overline{\boldsymbol{\sigma}_2}$ and $\overline{\boldsymbol{\sigma}_3}$

eigenstates represent equilateral tension or compression in the 12 isotropic plane, together with superposed tension along the 3 axis of symmetry of the medium.

Four strain eigentensors $\overline{\varepsilon_m}$, $m = 1, \ldots, 4$, correspond to these stress eigentensors $\overline{\boldsymbol{\sigma}_m}$, $m = 1, \ldots, 4$. Two of them, $\overline{\varepsilon_1}$ and $\overline{\varepsilon_4}$, are associated with pure distortion of the medium without any volume change. Hence, the elastic strain energy density T_d,

$$\begin{aligned} T_d &= T(\overline{\boldsymbol{\sigma}_1}) + T(\overline{\boldsymbol{\sigma}_4}) \\ &= (s_{11} - s_{12})[(\sigma_1 - \sigma_2)^2/2 + 2\sigma_6^2] + s_{44}(\sigma_4^2 + \sigma_5^2), \end{aligned} \quad (50)$$

owing to the contribution of the characteristic stress states $\overline{\boldsymbol{\sigma}_1}$ and $\overline{\boldsymbol{\sigma}_4}$, corresponds to a purely distortional strain energy. The remaining $\overline{\boldsymbol{\sigma}_2}$ and $\overline{\boldsymbol{\sigma}_3}$ stress tensors are associated solely with neither a distortional nor a dilatational form of strain energy; their respective tensors $\overline{\varepsilon_2}$ and $\overline{\varepsilon_3}$ cause both volume changes and shape distortions. Therefore, the elastic strain-energy density T_m, given by

$$\begin{aligned} T_m &= T(\overline{\boldsymbol{\sigma}_2}) + T(\overline{\boldsymbol{\sigma}_3}) \\ &= \big[\{[(s_{11} + s_{12})/2] + s_{33}/2\} + \big(\{[(s_{11} + s_{12})/2] - s_{33}/2\}^2 \\ &\quad + 2s_{13}^2\big)^{1/2}\big][-(\sin\omega/2^{1/2})(\sigma_1 + \sigma_2) + \cos\omega\sigma_3]^2 \\ &\quad + \big[\{[(s_{11} + s_{12})/2] + s_{33}/2\} - \big(\{[(s_{11} + s_{12})/2] - s_{33}/2\}^2 \\ &\quad + 2s_{13}^2\big)^{1/2}\big][(\cos\omega/2^{1/2})(\sigma_1 + \sigma_2) + \sin\omega\sigma_3]^2, \end{aligned} \quad (51)$$

and formed by stress eigentensors $\overline{\boldsymbol{\sigma}_2}$ and $\overline{\boldsymbol{\sigma}_3}$, corresponds to mixtures of distortional and dilatational components of elastic energy. Moreover, as seen by relation (50), the T_d strain-energy part is unrelated to the value of the eigenangle ω, whereas the strain-energy component T_m depends on the value of ω, which, thus, influences the type of strain energy stored in the body.

Interestingly, a direct geometric visualization for the stress eigentensors $\overline{\boldsymbol{\sigma}_m}$, $m = 1, \ldots, 4$, is obtained if we consider the projections of their contracted forms on the principal stress space $(\sigma_1, \sigma_2, \sigma_3)$. Accordingly, the characteristic stress state $\overline{\boldsymbol{\sigma}_4}$ vanishes at once, whereas the stress eigentensors $\overline{\boldsymbol{\sigma}_1}$, $\overline{\boldsymbol{\sigma}_2}$ and $\overline{\boldsymbol{\sigma}_3}$ are generally represented by three mutually orthogonal vectors, shown in Fig. 5, associated with the respective unit vectors $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$, which are defined by

$$\mathbf{e}_1 = [2^{-1/2}, -2^{-1/2}, 0]^T, \quad (52a)$$

$$\mathbf{e}_2 = [-2^{-1/2}\sin\omega, -2^{-1/2}\sin\omega, \cos\omega]^T, \quad (52b)$$

$$\mathbf{e}_3 = [2^{-1/2}\cos\omega, 2^{-1/2}\cos\omega, \sin\omega]^T. \quad (52c)$$

Then, it is noted in Fig. 5 that the unit vectors $\mathbf{e}_2$ and $\mathbf{e}_3$ are located on the principal diagonal plane (σ_3, δ_{12}), since they are equally inclined with respect to the principal stress axes σ_1 and σ_2. Furthermore, vectors $\mathbf{e}_2$ and $\mathbf{e}_3$ subtend angles equal to ω and $(\pi/2 - \omega)$, respectively, with the positive σ_3 axis. Finally, the $\mathbf{e}_1$ unit vector is perpendicular to the diagonal plane $\sigma_1 = \sigma_2$ and the σ_3 axis and, hence, lies on the deviatoric π plane. For any hexagonal medium, the direction of vector $\mathbf{e}_1$, and thus of eigentensor $\overline{\boldsymbol{\sigma}_1}$, remains constant, whereas vectors $\mathbf{e}_2$ and $\mathbf{e}_3$ rotate with respect to the origin of the coordinate frame, lying always on the plane $\sigma_1 = \sigma_2$.

5. Cubic symmetry

5.1. Cubic crystals

Cubic crystals are characterized by three different non-zero elastic compliances s_{11}, s_{44}, s_{12}. On that account, they may be regarded as special tetragonal-6 crystals, with $s_{16} = 0$ and $s_{11} = s_{33}$, $s_{12} = s_{13}$ and $s_{44} = s_{66}$. Considering in the following the spectral decomposition of the compliance tensor $\mathbf{S}$ for the class of cubic media, the Cartesian frame of reference is directed, with its 1 and 2 axes being axes of symmetry of the first order L. However, if this is not the case, the 1 axis coincides with the symmetry axis of the second-order L^2, while the 2 axis coincides with a symmetry axis of the second order, which is also an axis of complex symmetry of the fourth order, L_4^2. As a result, the eigenvalues λ_m, $m = 1, 2, 3$, of tensor $\mathbf{S}$ were obtained by reduction of relations (39), expressing the eigenvalues λ_m of the tetragonal-6 system, as follows:

$$\lambda_1 = s_{11} - s_{12}, \quad \lambda_2 = s_{44}/2, \quad \lambda_3 = s_{11} + 2s_{12}. \quad (53)$$

One characteristic value of cubic symmetry, namely λ_2, is identical to eigenvalue λ_4 of tetragonal-6 symmetry. In addition, eigenvalues λ_1 and λ_2 of tetragonal-6 media reduce to $s_{11} - s_{12}$, whereas eigenvalues λ_3 and λ_5 reduce to $s_{11} + 2s_{12}$ and $s_{44}/2$, respectively. Obviously, there are three distinct eigenvalues λ_m of tensor $\mathbf{S}$. One eigenvalue, λ_2, is of multiplicity three, eigenvalue λ_1 is of multiplicity two, and the last eigenvalue, λ_3, is of multiplicity one.

The eigenangle ω was also found for cubic symmetry, according to its defining relation (33), to be equal to 35.26°. Furthermore, the corresponding idempotent tensors $\mathbf{E}_1$ and $\mathbf{E}_2$ were derived from the defining equations (30) by substituting the value of the eigenangle ω and by adding, respectively, the first, $\mathbf{E}_1$, and second, $\mathbf{E}_2$, as well as the fourth, $\mathbf{E}_4$, and fifth, $\mathbf{E}_5$, idempotent tensors of tetragonal-6 media, while tensor $\mathbf{E}_3$ was obtained by reduction of the respective eigentensor of tetragonal-6 media:

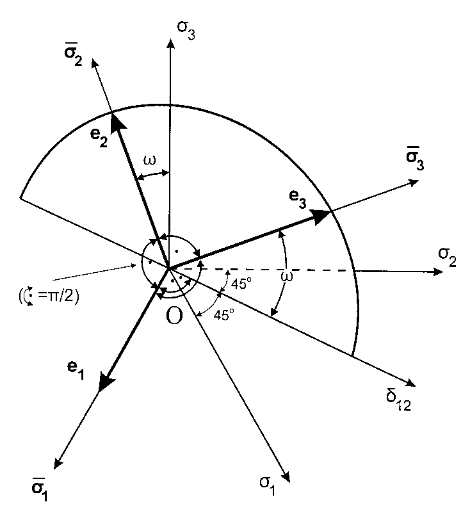

Figure 5
Geometric representation of the contracted stress eigentensors of the compliance fourth-rank tensor for hexagonal media in the principal stress space $(\sigma_1, \sigma_2, \sigma_3)$.

$$\mathbf{E}_1 = E^1_{ijkl} = \tfrac{1}{3}[(k_ik_j - l_il_j)(k_kk_l - l_kl_l) + (l_il_j - m_im_j)(l_kl_l - m_km_l) + (k_ik_j - m_im_j)(k_kk_l - m_km_l)], \quad (54a)$$

$$\mathbf{E}_2 = E^2_{ijkl} = \tfrac{1}{2}[(k_il_j + l_ik_j)(k_kl_l + l_kk_l) + (k_im_j + m_ik_j)(k_km_l + m_kk_l) + (l_im_j + m_il_j)(l_km_l + m_kl_l)], \quad (54b)$$

$$\mathbf{E}_3 = E^3_{ijkl} = \tfrac{1}{3}(\mathbf{1}\otimes\mathbf{1}) = \tfrac{1}{3}\delta_{ij}\delta_{kl}. \quad (54c)$$

Moreover, it appears by the form of expressions (54) that the idempotent tensors $\mathbf{E}_m$, obtained for the cubic symmetry *via* the spectral decomposition of the compliance tensor **S**, are exactly those derived by Walpole (1981, 1984), employing, however, altogether different types of decomposition of fourth-rank tensors.

In conclusion, the three distinct eigenvalues λ_m, $m = 1, 2, 3$, of the compliance fourth-rank tensor **S** constitute coordinate-invariant parameters, which characterize the elastic properties of cubic media. Moreover, the compliance tensor **S** is decomposed spectrally in terms of the characteristic values λ_m, defined according to relations (53), and the idempotent tensors $\mathbf{E}_m$ of (54), and it is valid that

$$\mathbf{S} = \lambda_1\mathbf{E}_1 + \lambda_2\mathbf{E}_2 + \lambda_3\mathbf{E}_3, \quad (55)$$

implying that the **L** space is decomposed into three subspaces $\mathbf{L}_m$, $m = 1, 2, 3$:

$$\mathbf{L} = \mathbf{L}_1 \oplus \mathbf{L}_2 \oplus \mathbf{L}_3, \quad (56)$$

where $\mathbf{L}_1$ is the subspace of deviators, corresponding to pure shear, $\dim(\mathbf{L}_1) = 2$, $\mathbf{L}_2$ is the subspace of deviators, corresponding to simple shear, $\dim(\mathbf{L}_2) = 3$ and $\mathbf{L}_3$ is the subspace of spherical tensors, $\dim(\mathbf{L}_3) = 1$.

Finally, the contracted characteristic stress tensors $\overline{\boldsymbol{\sigma}_m}$, $m = 1, 2, 3$, of the compliance tensor **S** were obtained from equations (44) by setting the value of the eigenangle ω equal to 35.26°. They are expressed as

$$\overline{\boldsymbol{\sigma}_1} = [\tfrac{1}{3}(2\sigma_1 - \sigma_2 - \sigma_3), \tfrac{1}{3}(-\sigma_1 + 2\sigma_2 - \sigma_3), \tfrac{1}{3}(-\sigma_1 - \sigma_2 + 2\sigma_3), 0, 0, 0]^T, \quad (57a)$$

$$\overline{\boldsymbol{\sigma}_2} = [0, 0, 0, \sigma_4, \sigma_5, \sigma_6]^T, \quad (57b)$$

$$\overline{\boldsymbol{\sigma}_3} = (3^{-1/2}\sigma_1 + 3^{-1/2}\sigma_2 + 3^{-1/2}\sigma_3)[3^{-1/2}, 3^{-1/2}, 3^{-1/2}, 0, 0, 0]^T. \quad (57c)$$

In consideration of relations (57), it is easily confirmed that these stress eigentensors $\overline{\boldsymbol{\sigma}_m}$ form an othogonal set and their sum equals the stress tensor $\boldsymbol{\sigma}$. Each of the three stress eigentensors $\overline{\boldsymbol{\sigma}_m}$ is associated with a unique type of loading. Eigenstate $\overline{\boldsymbol{\sigma}_1}$ is a pure shear stress state, stress eigentensor $\overline{\boldsymbol{\sigma}_2}$ refers to a simple shear stress state and, finally, the $\overline{\boldsymbol{\sigma}_3}$ eigentensor constitutes a hydrostatic loading.

Furthermore, three characteristic strain states $\overline{\varepsilon_m}$ are associated with stress eigentensors $\overline{\boldsymbol{\sigma}_m}$. Two of those are related with distortional alterations of the medium. Thus, the strain-energy density T_d, given by

$$T_d = T(\overline{\boldsymbol{\sigma}_1}) + T(\overline{\boldsymbol{\sigma}_2}) = (s_{11} - s_{12})[\tfrac{1}{2}(\sigma_1 - \sigma_2)^2 + \tfrac{1}{6}(-\sigma_1 - \sigma_2 + 2\sigma_3)^2] + s_{44}(\sigma_4^2 + \sigma_5^2 + \sigma_6^2), \quad (58)$$

and produced by $\overline{\boldsymbol{\sigma}_1}$ and $\overline{\boldsymbol{\sigma}_2}$, is a purely distortional strain energy. Conversely, the $\overline{\boldsymbol{\sigma}_3}$ stress state corresponds solely to a dilatational elastic energy T_v, given by

$$T_v = T(\overline{\boldsymbol{\sigma}_3}) = \tfrac{1}{3}(s_{11} + 2s_{12})(\sigma_1 + \sigma_2 + \sigma_3)^2. \quad (59)$$

In fact, relations (58) and (59) constitute the well known expressions for the distortional and dilatational energy components and the decomposition of the elastic potential T for the cubic medium resembles closely that of the isotropic body.

In addition, inasmuch as the projections of the contracted characteristic stress states $\overline{\boldsymbol{\sigma}_m}$, $m = 1, 2, 3$, given by relations (57), are concerned on the principal stress space $(\sigma_1, \sigma_2, \sigma_3)$, it is easily derived that the projection of the characteristic stress tensor $\overline{\boldsymbol{\sigma}_2}$, which is expressed as simple shear loading, becomes equal to zero. However, the projected stress eigenstates $\overline{\boldsymbol{\sigma}_1}$ and $\overline{\boldsymbol{\sigma}_3}$ are represented by a tri-orthogonal frame of vectors, corresponding to the unit vectors $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$ with the following definitions:

$$\mathbf{e}_1 = [2^{-1/2}, -2^{-1/2}, 0]^T, \quad (60a)$$

$$\mathbf{e}_2 = [-6^{-1/2}, -6^{-1/2}, 2(6^{-1/2})]^T, \quad (60b)$$

$$\mathbf{e}_3 = [3^{-1/2}, 3^{-1/2}, 3^{-1/2}]^T. \quad (60c)$$

Then, it is easily noted in Fig. 6 and ascertained by relations (60) that the unit vector $\mathbf{e}_3$ lies in the positive direction of the hydrostatic axis $(\sigma_1 = \sigma_2 = \sigma_3)$, whereas the unit vectors $\mathbf{e}_1$ and $\mathbf{e}_2$ lie on the deviatoric π plane, and both vectors $\mathbf{e}_2$ and $\mathbf{e}_3$ remain on the principal diagonal plane (σ_3, δ_{12}).

5.2. Isotropic crystals

In the well known case of isotropic elastic media, there are only two distinct matrix components s_{ij} in the Voigt notation,

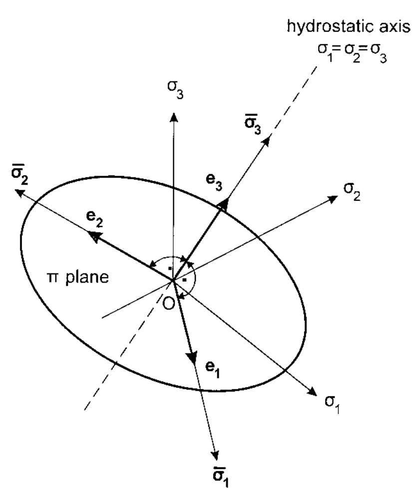

Figure 6
Representation of the characteristic stress states of the compliance fourth-rank tensor for both cubic and isotropic crystals in the principal stress space $(\sigma_1, \sigma_2, \sigma_3)$.

since it is also valid that $s_{44} = 2(s_{11} - s_{12})$. Consequently, it was found, by reduction of the expressions obtained for cubic symmetry, that one eigenvalue, λ_1, of the isotropic compliance tensor **S** is of multiplicity five, and one eigenvalue, λ_2, is of multiplicity one:

$$\lambda_1 = s_{44}/2, \quad \lambda_2 = s_{11} + 2s_{12} \tag{61}$$

and the values of the eigenangles are exactly those of cubic symmetry. The corresponding idempotent fourth-rank tensors $\mathbf{E}_m$, $m = 1, 2$, are expressed by

$$\mathbf{E}_1 = E^1_{ijkl} = \mathbf{I} - \tfrac{1}{3}(\mathbf{1} \otimes \mathbf{1}) \tag{62a}$$

$$\mathbf{E}_2 = E^2_{ijkl} = \tfrac{1}{3}(\mathbf{1} \otimes \mathbf{1}) = \tfrac{1}{3}\delta_{ij}\delta_{kl}, \tag{62b}$$

where **I** is the unit element of the fourth-rank tensor space.

In conclusion, the two distinct eigenvalues λ_m, $m = 1, 2$, of the isotropic material compliance tensor **S** constitute coordinate-invariant parameters, characterizing the elastic properties of isotropic bodies. Moreover, the compliance tensor **S** is decomposed spectrally according to the following representation:

$$\mathbf{S} = \lambda_1 \mathbf{E}_1 + \lambda_2 \mathbf{E}_2. \tag{63}$$

Then, the **L** space is decomposed into two subspaces $\mathbf{L}_m$, $m = 1, 2$:

$$\mathbf{L} = \mathbf{L}_1 \oplus \mathbf{L}_2, \tag{64}$$

where $\mathbf{L}_1$ is the five-dimensional subspace of deviators and $\mathbf{L}_2$ is the one-dimensional subspace of spherical tensors. Finally, the stress eigentensors for isotropy decompose an arbitrary loading $\boldsymbol{\sigma}$ as a sum of a deviatoric, $\overline{\boldsymbol{\sigma}_1}$, and a spherical, $\overline{\boldsymbol{\sigma}_2}$, tensor:

$$\boldsymbol{\sigma} = \overline{\boldsymbol{\sigma}_1} + \overline{\boldsymbol{\sigma}_2}, \quad \overline{\boldsymbol{\sigma}_2} = \tfrac{1}{3}(\operatorname{tr}\boldsymbol{\sigma})\mathbf{1}, \tag{65}$$

in which $\operatorname{tr}\boldsymbol{\sigma}$ is the first invariant of the stress $\boldsymbol{\sigma}$ tensor and the unit tensor **1** in Cartesian form is represented by the Kronecker delta. As a result, the total elastic strain-energy density T for the isotropic body is decomposed into distortional, T_d, and dilatational, T_v, components, as follows:

$$\begin{aligned} T &= T_d + T_v \\ &= T(\overline{\boldsymbol{\sigma}_1}) + T(\overline{\boldsymbol{\sigma}_2}) \\ &= (s_{44}/2)[\tfrac{1}{2}(\sigma_1 - \sigma_2)^2 + \tfrac{1}{6}(-\sigma_1 - \sigma_2 + 2\sigma_3)^2 \\ &\quad + 2(\sigma_4^2 + \sigma_5^2 + \sigma_6^2)] + \tfrac{1}{3}(s_{11} + 2s_{12})(\sigma_1 + \sigma_2 + \sigma_3)^2. \end{aligned} \tag{66}$$

6. Thermodynamically admissible compliance-tensor components

Within the context of the classical linear anisotropic elasticity, accordance with thermodynamics requires the strain-energy function T to be positive definite, expressed as a quadratic form, in either the stress, $\boldsymbol{\sigma}$, or the strain, ε, tensor, by

$$2T = \boldsymbol{\sigma} \cdot \mathbf{S} \cdot \boldsymbol{\sigma} = S_{ijkl}\sigma_{ij}\sigma_{kl} > 0 \tag{67a}$$

$$2T = \varepsilon \cdot \mathbf{C} \cdot \varepsilon = C_{ijkl}\varepsilon_{ij}\varepsilon_{kl} > 0, \tag{67b}$$

for any non-zero symmetric second-rank tensor $\boldsymbol{\sigma}$ or ε. In the case of the spectral decomposition of either the compliance, **S**, or stiffness, **C**, fourth-rank tensors, the problem reduces to the trivial conditions of non-negativeness of the eigenvalues λ_m of the corresponding tensors, according to the conditions

$$\lambda_m > 0, \quad m = 1, \ldots, k. \tag{68}$$

Therefore, the positiveness of the eigenvalues λ_m of the compliance tensor **S** exhibiting any symmetry allows the determination of necessary and sufficient conditions for the positive definiteness of T in terms of the compliance-tensor components S_{ijkl}, as expressed in the Voigt notation. The resulting conditions are equivalent to the expressions involving the positivity of the leading principal minors of the square matrix associated with tensor **S**, usually employed in the literature (Ting, 1996).

By imposing the eigenvalues λ_m of the compliance tensor **S** of orthorhombic media, expressed by relations (11), to be positive definite, it is readily deduced that

$$A < 0, \quad k > 0, \quad Q^2 < 4k^3. \tag{69}$$

Two inequalities follow directly from condition $k > 0$, that is $A < -(3B)^{1/2}$ and $A > (3B)^{1/2}$, yet the second one is excluded in view of the condition $A < 0$. Only the first case, $A < -(3B)^{1/2}$, is admissible; still, it is required that $B > 0$. Further, it is valid that

$$4k^3 - Q^2 = A^2/27(B^2 - 4AC) - 2B/3(2B^2/9 - AC) - C^2$$

and, hence, in order for the condition $Q^2 < 4k^3$ to be satisfied, it is requested that

$$0 < 2B^2/9 < AC < B^2/4, \tag{70}$$

which is evidently fulfilled only when $C < 0$.

Therefore, the positivity of eigenvalues λ_1, λ_2, λ_3 is equivalent to $A < 0$, $B > 0$ and $C < 0$. In fact, these inequalities would have easily been obtained, alternatively, noting that the coefficients A, B and C of the cubic polynomial (3) in λ are equal to

$$A = -(\lambda_1 + \lambda_2 + \lambda_3), \quad B = \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_1\lambda_3, \quad C = -\lambda_1\lambda_2\lambda_3. \tag{71}$$

Consequently, for λ_1, λ_2, λ_3 to be positive, it is imperative that the ensuing inequalities be satisfied simultaneously:

$$s_{11} + s_{22} + s_{33} > 0,$$

$$s_{11}s_{22} + s_{22}s_{33} + s_{11}s_{33} - (s_{12}^2 + s_{13}^2 + s_{23}^2) > 0, \tag{72a}$$

$$s_{11}(s_{22}s_{33} - s_{23}^2) - s_{12}(s_{12}s_{33} - s_{13}s_{23}) + s_{13}(s_{12}s_{23} - s_{13}s_{22}) > 0. \tag{72b}$$

Concerning the first inequality of the set (72*a*), it is noted that the elastic compliances s_{11}, s_{22} and s_{33} are all positive or any two of them are positive and one is negative or, finally, two of them are negative and the remaining one is positive. Considering the third case, $s_{11} > 0$ and $\{s_{22}, s_{33}\} < 0$ gives $s_{11} > -(s_{22} + s_{33}) > 0$. Then, it is valid that $s_{11} > -s_{22}$ and $-(s_{22} + s_{33}) > -s_{33}$ and, hence, we have that

$$-s_{11}(s_{22}+s_{33}) > s_{22}s_{33} > 0. \quad (73)$$

However, noting that $s_{12}^2 + s_{13}^2 + s_{23}^2 > 0$, it follows from the second inequality of (72*a*) that $s_{22}s_{33} > -s_{11}(s_{22}+s_{33}) > 0$, suggesting that the case of one compliance being positive and two negative is not compatible with the initial two inequalities of the set (72*a*). It is, similarly, shown that the case of two positive compliances and a negative one cannot satisfy relations (72*a*), and on that account it is concluded that the latter conditions are met fully only when $\{s_{11}, s_{22}, s_{33}\} > 0$.

Finally, as far as the second relation in (72*a*) is concerned, this leaves us with three choices. The differences $s_{11}s_{22} - s_{12}^2$, $s_{22}s_{33} - s_{23}^2$ and $s_{11}s_{33} - s_{13}^2$ are all positive or two of them are positive and one is negative or two of them are negative and one is positive. It is next shown that only the first case of those will satisfy equation (72*b*). Since $\{s_{11}s_{23}^2, s_{22}s_{13}^2, s_{33}s_{12}^2\} > 0$, the latter inequality gives

$$s_{11}s_{22}s_{33} > s_{12}s_{13}s_{23}. \quad (74)$$

Then, considering the second case, it follows that $(s_{11}s_{22}^2s_{33})^{1/2} > s_{12}s_{23}$, since $(s_{11}s_{22})^{1/2} > s_{12} > -(s_{11}s_{22})^{1/2}$ and $(s_{22}s_{33})^{1/2} > s_{23} > -(s_{22}s_{33})^{1/2}$. Yet, $s_{11}s_{33} < s_{13}^2$, so that relation (74) is not clearly satisfied. This contradiction is only removed if the first case is considered, namely that $s_{11}s_{22} > s_{12}^2$, $s_{22}s_{33} > s_{23}^2$ and $s_{11}s_{33} > s_{13}^2$.

In conclusion, all the diagonal compliance elements, in the Cartesian coordinate frame with respect to which the compliance-tensor components were defined by relations (1), must be positive. Moreover, it is seen that the values of the off-diagonal elements are bounded by certain inequalities. In summary, the following set of constraints must be satisfied by the compliance-tensor components:

$$\{s_{11}, s_{22}, s_{33}, s_{44}, s_{55}, s_{66}\} > 0, \quad (75a)$$

$$s_{12}^2 < s_{11}s_{22}, \quad s_{13}^2 < s_{11}s_{33}, \quad s_{23}^2 < s_{22}s_{33}, \quad (75b)$$

$$s_{11}(s_{22}s_{33} - s_{23}^2) - s_{12}(s_{12}s_{33} - s_{13}s_{23}) + s_{13}(s_{12}s_{23} - s_{13}s_{22}) > 0. \quad (75c)$$

Besides, it was proven for the class of tetragonal-7 media that constraint (68) requests the subsequent set of conditions:

$$\{s_{11}, s_{33}, s_{44}, s_{66}\} > 0, \quad (76a)$$

$$s_{12}^2 < s_{11}^2, \quad s_{13}^2 < s_{11}s_{33}, \quad s_{16}^2 < s_{11}s_{66}, \quad (76b)$$

$$(s_{11} - s_{12})s_{66} > 2s_{16}^2, \quad (s_{11} + s_{12})s_{33} > 2s_{13}^2. \quad (76c)$$

In addition, the restrictive bounds, which should be satisfied by the thermodynamically admissible compliance-tensor components of tetragonal-6 symmetry, are obtained by reduction of equations (76), noting that in this particular situation $s_{16} = 0$. Thus, the following inequalities are found:

$$\{s_{11}, s_{33}, s_{44}, s_{66}\} > 0, \quad (77a)$$

$$s_{12}^2 < s_{11}^2, \quad s_{13}^2 < s_{11}s_{33}, \quad (s_{11} + s_{12})s_{33} > 2s_{13}^2. \quad (77b)$$

Therefore, relations (76) and (77) may be used for the qualification of the experimentally measured elastic compliances of tetragonal crystals.

For the special case of transverse isotropy or hexagonal symmetry, in which the elastic properties in the 12 plane are identical for all directions about the normal, the constraints of equation (77) are reduced to

$$\{s_{11}, s_{33}, s_{44}\} > 0, \quad (78a)$$

$$s_{12}^2 < s_{11}^2, \quad s_{13}^2 < s_{11}s_{33}, \quad (s_{11} + s_{12})s_{33} > 2s_{13}^2. \quad (78b)$$

Similar bounds were first obtained by Eubanks & Sternberg (1954) and later by Nye (1957) and Lempriere (1968), pursuing different but mathematically equivalent methods. Christensen (1979) also attempted to derive such bounds on the basis of physical arguments, however, the bounds he derived were less stringent, overestimating the interval of values of s_{13}, which was assumed to be

$$s_{13}^2 < s_{11}s_{33}. \quad (79)$$

Then, comparison of the constraint obtained by Christensen with the one expressed by the right-hand side of relation (78) shows that inequality (79) is exact only in the limiting case, $s_{12} = s_{11}$. Finally, Theocaris & Philippidis (1991) also derived inequalities (78) by utilizing the same analysis as the one presented in this paper, *via* the spectral decomposition of tensor **S** for transversely isotropic media.

Furthermore, the restrictions applying to the elastic compliances for cubic media are obtained by reduction of the general conditions (77), which are valid for tetragonal-6 media, and are expressed by

$$\{s_{11}, s_{44}\} > 0, \quad s_{11} > s_{12}, \quad s_{11} + 2s_{12} > 0. \quad (80)$$

These constraints for cubic media were first stated by Nye (1957). At last, for the isotropic case of equal elastic properties in all directions within the medium, conditions (80) simplify further to the well known constraints:

$$s_{11} > 0, \quad -1 < -s_{12}/s_{11} < \tfrac{1}{2}. \quad (81)$$

7. Conclusions

In this paper, the cases of orthorhombic, tetragonal, hexagonal and cubic media have been treated in detail, and all the invariant parameters of the spectral decomposition of the compliance tensor were explicitly calculated in terms of its Cartesian components. Furthermore, the strain-energy density of the corresponding media was given a definite decomposition in distinct components, associated with the elastic stress and strain eigenstates. In fact, the decomposition of the strain-energy density for the orthorhombic and the tetragonal-7 media, as well as their subclasses, the tetragonal-6, hexagonal and cubic media, which is based on the spectral decomposition of the compliance and stiffness tensors of such media, constitutes the simplest one for anisotropic media. It was shown by reduction that it corresponds to the well known classical decomposition of the elastic strain-energy density of isotropic solids into dilatational and distortional forms of energy.

The elastic characteristics of anisotropic crystals are offered by invariant parameters, emerging from the spectral decomposition of its compliance tensor. Then, the characteristic values of the compliance tensor constitute these parameters, together with the eigenangles. In fact, the eigenangles were shown to determine the alignment of the eigentensors, associated with the eigenvalues of the compliance tensor, when represented in a stress coordinate system.

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